

# Review Handout For Math 2280

## Trigonometric Identities

$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$	$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$
$\sin^2 \theta = \frac{1}{2}[1 - \cos(2\theta)]$	$\cos^2 \theta = \frac{1}{2}[1 + \cos(2\theta)]$
$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$	$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$
$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$	$\sin x + \sin y = 2 \sin \left(\frac{x+y}{2}\right) \cos \left(\frac{x-y}{2}\right)$
$\cos x + \cos y = 2 \cos \left(\frac{x+y}{2}\right) \cos \left(\frac{x-y}{2}\right)$	$\cos x - \cos y = 2 \sin \left(\frac{x+y}{2}\right) \sin \left(\frac{y-x}{2}\right)$

Here,  $\omega$  is the lower case Greek letter omega.

$A \cos(\omega t) + B \sin(\omega t) = C \sin(\omega t + \phi)$  where  $C = \sqrt{A^2 + B^2}$  and  $\phi$  is the angle such that  $\sin \phi = \frac{A}{C}$  and  $\cos \phi = \frac{B}{C}$ , or  $\phi = \begin{cases} \tan^{-1}(\frac{A}{B}) & , \text{ if } B > 0 \\ \pi + \tan^{-1}(\frac{A}{B}) & , \text{ if } B < 0 \end{cases}$ .

Complex Exponential (Euler's Formula):  $e^{i\theta} = \cos \theta + i \sin \theta$

Polar Form of Complex Numbers:  $z = a + bi = r e^{i\theta} = r \cos \theta + i r \sin \theta$

where  $r = \sqrt{a^2 + b^2}$  and  $\theta$  is the angle such that  $\sin \theta = \frac{b}{r}$  and  $\cos \theta = \frac{a}{r}$ , or

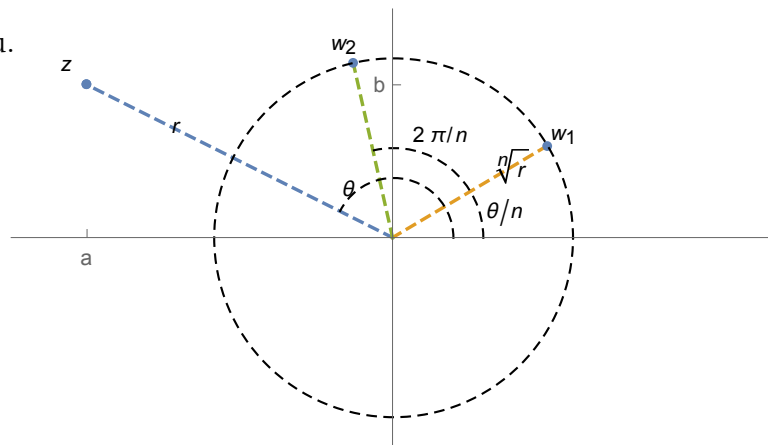
$$\theta = \begin{cases} \tan^{-1}(\frac{b}{a}) & , \text{ if } a > 0 \\ \pi + \tan^{-1}(\frac{b}{a}) & , \text{ if } a < 0 \end{cases}$$

De Moivre's Formula: For  $z = a + bi = r e^{i\theta}$ , its  $n$ th power is  $z^n = r^n e^{in\theta} = r^n \cos(n\theta) + i r^n \sin(n\theta)$ .

Here,  $w$  is the lower case of the letter double-u.

The  $n$   $n$ -th roots of  $z = a + bi = r e^{i\theta}$ , are  $w_k = \sqrt[n]{r} e^{i(\frac{\theta+2(k-1)\pi}{n})} = \sqrt[n]{r} \cos\left(\frac{\theta+2(k-1)\pi}{n}\right) + i \sqrt[n]{r} \sin\left(\frac{\theta+2(k-1)\pi}{n}\right)$  for  $k = 1, \dots, n$ .

Or, the solutions of  $w^n = a + bi = r e^{i\theta}$  are  $w_k = \sqrt[n]{r} e^{i(\frac{\theta+2(k-1)\pi}{n})} = \sqrt[n]{r} \cos\left(\frac{\theta+2(k-1)\pi}{n}\right) + i \sqrt[n]{r} \sin\left(\frac{\theta+2(k-1)\pi}{n}\right)$  for  $k = 1, \dots, n$ .



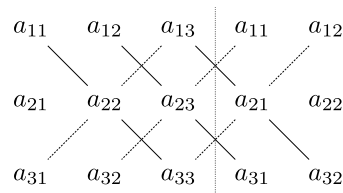
In words, the  $n$   $n$ -th roots of  $r e^{i\theta}$  are on a circle of radius  $\sqrt[n]{r}$  and are  $\frac{2\pi}{n}$  radians apart with the first one having the angle  $\frac{\theta}{n}$ .

Determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The determinant of the three columns on the left is the sum of the products along the solid diagonals minus the sum of the products along the dashed diagonals. \*



For matrix  $A = \begin{bmatrix} a_{11} & \dots & a_{1(j-1)} & a_{1j} & a_{1(j+1)} & \dots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & \dots & a_{(i-1)(j-1)} & a_{(i-1)j} & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ a_{i1} & \dots & a_{i(j-1)} & a_{ij} & a_{i(j+1)} & \dots & a_{in} \\ a_{(i+1)1} & \dots & a_{(i+1)(j-1)} & a_{(i+1)j} & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{n(j-1)} & a_{nj} & a_{n(j+1)} & \dots & a_{nn} \end{bmatrix}$ , let  $A_{ij}$

be the matrix with the  $i$ th row and  $j$ th column removed:

$$A_{ij} = \begin{bmatrix} a_{11} & \dots & a_{1(j-1)} & a_{1(j+1)} & \dots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & \dots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ a_{(i+1)1} & \dots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{n(j-1)} & a_{n(j+1)} & \dots & a_{nn} \end{bmatrix}.$$

Then, for any row  $i$ ,  $\text{Det}A = |A| = \sum_{j=1}^n (-1)^{i+j} a_{ij} |A_{ij}|$ .

Also, for any column  $j$ ,  $\text{Det}A = |A| = \sum_{i=1}^n (-1)^{i+j} a_{ij} |A_{ij}|$ .

\* The picture in the right is from Wikipedia.

### Differentiation Rules

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) \quad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = (\ln a) a^x$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x}$$

$$\frac{d}{dx} \log_a x = \frac{1}{(\ln a)x}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

Fundamental Theorem of Calculus: For a continuous function  $f$ ,  $\frac{d}{dx} \left[ \int_a^x f(t) dt \right] = f(x)$

### Strategies For Integration

Method	Example
Expanding	$(e^x - e^{-x})^2 = e^{2x} - 2 + e^{-2x}$
Completing the square	$\sqrt{8x - x^2} = \sqrt{16 - (x - 4)^2}$
Using a trigonometric identity	$\sin^2 x = \frac{1}{2} [1 - \cos(2x)]$
Eliminating a square root	$\sqrt{x^2 + 2 + \frac{1}{x^2}} = \sqrt{\left(x + \frac{1}{x}\right)^2} = \left x + \frac{1}{x}\right $
Reducing an improper fraction	$\frac{x^3 - 7x}{x - 2} = x^2 + 2x - 3 - \frac{6}{x - 2}$
Separating a fraction	$\frac{3x + 2}{1 - x^2} = \frac{3x}{1 - x^2} + \frac{2}{1 - x^2}$
Multiplying by a form of 1	$\sec x = \sec x \times \frac{\sec x + \tan x}{\sec x + \tan x} = \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$

## Integration Techniques

Method	Example
<p><math>u</math>-substitution</p> $\int f(g(x))g'(x)dx = \int f(u)du$	<p>For <math>u = x^2 + 1</math> we have <math>x dx = \frac{du}{2}</math> and so</p> $\int \frac{x}{\sqrt{x^2 + 1}} dx = \int \frac{du}{2\sqrt{u}}$
<p>Integration by parts</p> $\int u dv = uv - \int v du$	<p>For <math>u = x</math>, <math>dv = e^x dx</math> we have  <math>du = dx</math>, <math>v = e^x</math> and so</p> $\int x e^x dx = x e^x - \int e^x dx$
<p>Trigonometric Substitution</p> <p>For expressions <math>\pm x^2 \pm a^2</math></p>	<p>For <math>x = \tan \theta</math> with <math>-\pi/2 &lt; \theta &lt; \pi/2</math>  we have <math>\sec \theta &gt; 0</math>, <math>dx = \sec^2 \theta d\theta</math> and <math>\sqrt{1 + x^2} = \sqrt{1 + \tan^2 \theta} = \sqrt{\sec^2 \theta} =  \sec \theta  = \sec \theta</math></p> <p>Thus <math>\int \frac{dx}{\sqrt{1 + x^2}} = \int \frac{\sec^2 \theta}{\sec \theta} d\theta</math></p>
<p>Partial Fractions</p> <p><math>\int \frac{P(x)}{D(x)} dx</math> where <math>P</math> and <math>D</math> are polynomials with <math>\deg P &lt; \deg D</math>.  Factor <math>D(x)</math>.</p>	$\int \frac{dx}{x^3 + x^2} = \int \frac{dx}{x^2(x + 1)}$ $= \int \left( -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x + 1} \right) dx$

## Table of Integration Formulas

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ , for $n \neq -1$	$\int \frac{1}{x} dx = \ln  x  + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int \sin x dx = -\cos x + C$	$\int \cos x dx = \sin x + C$
$\int \sec^2 x dx = \tan x + C$	$\int \csc^2 x dx = -\cot x + C$
$\int \sec x \tan x dx = \sec x + C$	$\int \csc x \cot x dx = -\csc x + C$
$\int \sec x dx = \ln  \sec x + \tan x  + C$	$\int \csc x dx = \ln  \csc x - \cot x  + C$
$\int \tan x dx = \ln  \sec x  + C$	$\int \cot x dx = \ln  \sin x  + C$
$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C$	$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left( \frac{x}{a} \right) + C$
$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left  \frac{x - a}{x + a} \right  + C$	$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln  x + \sqrt{x^2 \pm a^2}  + C$
$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left( \frac{x}{a} \right) + C$	$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln \left( x + \sqrt{x^2 + a^2} \right) + C$

For the power series  $\sum_{n=0}^{\infty} c_n(x - x_0)^n$  exactly one of the following three cases will hold and for each case a radius of convergence  $\rho$  is defined.

1.  $\sum_{n=0}^{\infty} c_n(x - x_0)^n$  converges only for  $x = x_0$  and  $\rho = 0$ .
2.  $\sum_{n=0}^{\infty} c_n(x - x_0)^n$  converges for all  $x$  and  $\rho = \infty$ .
3.  $\sum_{n=0}^{\infty} c_n(x - x_0)^n$  converges if  $|x - x_0| < R$  and diverges if  $|x - x_0| > R$ , for some positive number  $R$ , and  $\rho = R$ .

The radius of convergence and the interval of convergence of any power series  $\sum_{n=0}^{\infty} c_n(x - x_0)^n$  can be

found as follows. Let  $a_n = c_n(x - x_0)^n$  and consider  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ .

1. If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$  holds only for  $x = x_0$ , then  $\rho = 0$  and the interval of convergence is  $\{x_0\}$ .
2. If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$  holds for all  $x$ , then  $\rho = \infty$  and the interval of convergence is  $(-\infty, \infty)$ .
3.  $\rho > 0$  is the radius of convergence if  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$  holds for  $|x - x_0| < \rho$  and the interval of convergence is  $(x_0 - \rho, x_0 + \rho)$  plus none, one or both endpoints  $x = x_0 \pm \rho$  which must be checked individually.

The above method works for **any** power of the term  $(x - x_0)$  in the power series, **not just  $n$** .

**I do not recommend using the following method stated in your textbook since it will not work in every case, for example, it fails if the series is of the form  $\sum_{n=0}^{\infty} c_n(x - x_0)^{2n}$ .**

However, for sake of completeness, I have included it.

For only the power series of the form  $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ , the radius of convergence can be found as follows. If,  $c_n \neq 0$  for large  $n$ , and  $\lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right| = L$ , where  $0 \leq L \leq \infty$ , then  $\rho = L$ . In this version, for  $L$  positive and finite, the interval of convergence is  $(x_0 - L, x_0 + L)$  plus none, one or both endpoints  $x = x_0 \pm L$  which must be checked individually.